Quantum Field Theory in Condensed Matter Physics  
Summer Semester 2010/11  

Problem Set 10: BCS Theory  
(Tutorial 04.07.2011, Peter Machon)

Problem 1 – Equation of motion for the BCS Green’s function

The BCS Hamiltonian and the particle number operator are given by
\[\hat{H} = \int \left[ \sum_\alpha -\frac{\hat{\psi}_\alpha^\dagger \hat{\psi}_\alpha}{2m} \nabla^2 + \frac{g}{2} \sum_{\alpha,\beta} \hat{\psi}_\alpha^\dagger \hat{\psi}_\beta \hat{\psi}_\alpha \hat{\psi}_\beta \right] d^3r \quad \hat{N} = \sum_\alpha \int \hat{\psi}_\alpha^\dagger \hat{\psi}_\alpha d^3r \]  
where \(g\) denotes the interaction strength.

(a) Eq. 1 defines Heisenberg operators given by
\[\tilde{\psi}_\alpha(\vec{r}, \tau) = e^{(\hat{H} - \mu \hat{N})\tau} \psi_\alpha(\vec{r}) e^{-(\hat{H} - \mu \hat{N})\tau} \quad \tilde{\psi}_\alpha(\vec{r}, \tau) = e^{(\hat{H} - \mu \hat{N})\tau} \psi_\alpha^\dagger(\vec{r}) e^{-(\hat{H} - \mu \hat{N})\tau} \]

Since \(\hat{H}\) and \(\hat{N}\) commute, show that the e.o.m. of the Green’s function \(G_{\alpha\beta}(x_1, x_2) = -\left\langle T\tilde{\psi}_\alpha(x_1)\tilde{\psi}_\beta(x_2) \right\rangle\) is given by
\[\left(\frac{\partial G_{\alpha\beta}(x_1, x_2)}{\partial \tau_1}\right) = -\delta_{\alpha\beta}\delta(x_1 - x_2) + \left(\frac{\nabla^2_1}{2m} + \mu\right) G_{\alpha\beta}(x_1, x_2) + g \left\langle T\tilde{\psi}_\gamma(x_1)\tilde{\psi}_\alpha(x_1)\tilde{\psi}_\alpha(x_1)\tilde{\psi}_\beta(x_2) \right\rangle \]  

Hint: Calculate first the time derivative of \(\tilde{\psi}_\alpha(x) \quad (x = (\vec{r}, \tau))\) like \(\partial_\tau \tilde{\psi}_\alpha = \left[ \hat{H} - \mu \hat{N}, \tilde{\psi}_\alpha \right]\).

(b) Now we will use Wick’s theorem to simplify the interaction part of the e.o.m. The basic assumption of the BCS theory is that the terms
\[F_{\alpha\beta}(x_1, x_2) = -\left\langle T\tilde{\psi}_\alpha(x_1)\tilde{\psi}_\beta(x_2) \right\rangle \quad F_{\alpha\beta}^\dagger(x_1, x_2) = -\left\langle T\tilde{\psi}_\alpha^\dagger(x_1)\tilde{\psi}_\beta^\dagger(x_2) \right\rangle \]
(called anomalous Gorkov or Green’s functions) do not vanish for a macroscopic volume \((V \rightarrow \infty)\) since the electrons form Cooper pairs which condense into a ground state. Additionally are all the other terms of the interaction part ignored in BCS theory since they lead to a renormalisation of the chemical potential \(\mu\). Define the gap matrix like \(\Delta_{\alpha\beta}(x) = |g|F_{\alpha\beta}(x, x)\) and show that the e.o.m. becomes \((g < 0)\):
\[\left(\frac{\partial}{\partial \tau_1} - \frac{\nabla^2_1}{2m} - \mu\right) G_{\alpha\beta}(x_1, x_2) - \Delta_{\alpha\gamma}(x_1)F_{\gamma\beta}^\dagger(x_1, x_2) = -\delta_{\alpha\beta}\delta(x_1 - x_2) \]
Problem 2 – Gorkov equation for a homogeneous system

Proceeding like in Problem 1 to find an e.o.m for $F^\dagger$, repeating everything for the time-reversed Green’s function $\tilde G_{\alpha\beta}(x_1, x_2) = \left\langle T \psi^\dagger_\alpha(x_1) \tilde \psi_\beta(x_2) \right\rangle = \delta_{\alpha\beta} \tilde G(x_1, x_2) = \delta_{\alpha\beta} G(x_2, x_1)$ and introducing the Nambu space like in the lecture one finally ends up with the matrix Green’s function for BCS superconductors:

$$\tilde G(x_1, x_2) = \begin{pmatrix} G(x_1, x_2) & F(x_1, x_2) \\ -F^\dagger(x_1, x_2) & \tilde G(x_1, x_2) \end{pmatrix}$$ (4)

With the operator $\tilde G^{-1} = -\frac{\partial}{\partial \tau} - \tilde H$ the Gorkov eqn. can be written as $\tilde G^{-1}(x_1) \tilde G(x_1, x_2) = 1 \delta(x_1 - x_2)$ were we defined

$$\tilde H = \begin{pmatrix} -\frac{\nabla^2}{2m} - \mu & -\Delta \\ -\Delta^* & \frac{\nabla^2}{2m} + \mu \end{pmatrix}$$ (5)

Introduce the Matsubara frequency representation and assume the system to be homogeneous to go over to k space and show that the matrix Green’s function in Nambu space is

$$\tilde G = \frac{-i\omega_n - \varepsilon_k \tilde \tau_3 + \Delta}{\omega_n^2 + \varepsilon_k^2 + \Delta^2}$$ (6)

were $\Delta = \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix}$ and $\varepsilon_k = k^2/2m - \mu$.

Problem 3 – Anderson theorem

One important problem is impurity scattering in the superconductor (so called dirty superconductors). The Anderson theorem tells that this impurity scattering doesn’t effect the superconducting gap $\Delta$.

(a) To derive the Anderson effect first evaluate the integral in the self energy by inserting the Green’s function of problem 2 as the unperturbed Green’s function $\tilde G_0$ and show that

$$\tilde \Sigma = \frac{1}{2\pi \tau_{imp}} \int_{-\infty}^{\infty} d\varepsilon_k \tilde \tau_3 \tilde G_0 \tilde \tau_3 = \frac{1}{2\tau_{imp}} \left( \frac{-i\omega_n}{\sqrt{\omega_n^2 + \Delta^2}} - \frac{\Delta}{\sqrt{\omega_n^2 + \Delta^2}} \right)$$ (7)

This self energy takes into account all impurity diagrams of the type shown in the picture.

(b) Insert the impurity self energy in the Dyson equation to show that

$$\tilde G^{-1} = i\omega_n - \varepsilon_k \tilde \tau_3 + \Delta$$ (8)

were $\tilde a = a(1 + 1/2\tau_{imp} \Omega_n)$ and $\Omega_n = \sqrt{\omega_n^2 + \Delta^2}$

(c) Insert the new $F$ from (b) in the definition of $\tilde \Delta$ to show that the resulting gap matrix is equal to the unperturbed one that was introduced in the lecture.