Quantum Field Theory in Condensed Matter Physics
Summer Semester 2011

Problem Set 4
(16.5.2011, Cecilia Holmqvist)

Problem 1

(a) Show that

\[
\int dr'' V(r - r'') \langle T \left( \bar{\psi}(r'', \tau) \psi(r', \tau) \psi(r, \tau) \bar{\psi}(r', \tau') \right) \rangle = \int dr'' V(r - r'') \lim_{\tau_2 \to \tau_1 \to \tau} G(r'', r_1; r, \tau; r', \tau'; r'', \tau_2),
\]

where \( G(r_1, \tau_1; r_2, \tau_2; r_3, \tau_3; r_4, \tau_4) = \langle T \bar{\psi}(r_1, \tau_1) \psi(r_2, \tau_2) \bar{\psi}(r_4, \tau_4) \bar{\psi}(r_3, \tau_3) \rangle \).

(b) Compare the temperature, retarded and time-ordered Green’s functions for free Fermions,

\[
G_k(\tau) = -\langle T(\bar{c}_k(\tau)c_k^\dagger) \rangle, \\
G^R_k(t) = -i\theta(t)\langle\{c_k(t), c_k^\dagger\}\rangle \\
and \ G^T_k(t) = -i\langle T(c_k(t)c_k^\dagger) \rangle,
\]

and discuss their information content.

Problem 2 – Wick’s theorem

Consider a system of non-interacting Fermions with Hamiltonian \( H = \sum_\alpha \epsilon_\alpha a^\dagger_\alpha a_\alpha \) and the expectation value of a time-ordered product of Fermionic operators

\[
S = \langle T(c(\tau_n)c(\tau_{n-1}) \ldots c(\tau_2)c(\tau_1)) \rangle = \text{Tr}(\rho_T T(c(\tau_n)c(\tau_{n-1}) \ldots c(\tau_2)c(\tau_1)))
\]

where \( c \) may be a creation (\( a^\dagger \)) or annihilation (\( a \) ) operator, and

\[
\rho_T = \frac{e^{-H/kT}}{\text{Tr}e^{-H/kT}}
\]
is the statistical operator for the equilibrium state. For simplicity, assume that the operators in eq. 3 already are time ordered (otherwise one can relabel the terms at the cost of introducing an overall sign factor \((-1)^P\) depending on the number of permutations \(P\)).

(a) Show that for fermions

\[
\langle \{c_q,A\} \rangle = (1 + e^{\lambda_c q / kT}) \langle c_q A \rangle,
\]

where \(A\) is an operator and \(\lambda_c = 1 (-1)\) if \(c_q\) is a creation (annihilation) operator.

(b) Show that the commutator \(\{c_q(\tau_m), c_{q'}(\tau_n)\}\) is a complex number,

\[
\{c_q(\tau_m), c_{q'}(\tau_n)\} = \delta_{q,q'} (1 + e^{\lambda_c q / kT}) \langle c_q(\tau_m)c_{q'}(\tau_n) \rangle.
\]

(c) Show that

\[
S = \langle c(\tau_{2N}) c(\tau_{2N-1}) \ldots c(\tau_2) c(\tau_1) \rangle
\]

\[
= \sum_{n=1}^{2N-1} \langle c(\tau_{2N}) c(\tau_n) \rangle \langle \prod_{m=1, m \neq n}^{2N-1} c(\tau_m) \rangle.
\]

Write \(S\) as \(S = \langle \prod_{n=1}^{2N} c(\tau_n) \rangle = \langle c(\tau_{2N}) \prod_{n=1}^{2N-1} c(\tau_n) \rangle\) and use the result of (a) and (b). (Remember that the number of operators has to be even, otherwise one of the expectation values in the product will be zero.) Then, by induction, \(S\) can be written as a sum over all possible pairs.

**Problem 3 (Bonus)**

Repeat the problems 1 and 2 for bosons.