Theoretische Festkörperphysik
Sommersemester 2012
Übungsblatt 8
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Exercise 1 – Landau-Levels: Algebraic method and gauge invariance

We consider a two-dimensional electron gas with electron mass $M$ in a perpendicular magnetic field $B_z = \partial_x A_y - \partial_y A_x$ (in the case of a 3DEG, the situation is essentially identical, since the additional term $\frac{p_z^2}{2M}$ is trivially incorporated).

In addition to the canonically momentum $\hat{p} = (\hat{p}_x, \hat{p}_y)$, $\hat{r} = (\hat{x}, \hat{y})$ with $[\hat{x}, \hat{p}_i] = i\hbar$, one introduces the gauge invariant (or kinetic) momentum $\hat{\pi} = \hat{p} - qA$, so that the Hamiltonian can then be written as

$$H = \frac{\hat{\pi}^2}{2M}$$

with the stationary eigenvalue equation $H|\Psi_n(x,y)\rangle = E_n|\Psi_n(x,y)\rangle$.

(a) How does the wave-function $|\Psi(x,y)\rangle$ change under the gauge-transformation $A' = A + \nabla \chi$ with the arbitrary gauge field $\chi(x,y)$. Show that the expectation value of the kinetic momentum is gauge invariant, i.e. verify that $\langle \Psi(x,y)|\hat{\pi}|\Psi(x,y)\rangle = \langle \Psi'(x,y)|\hat{\pi}'|\Psi'(x,y)\rangle$. Which of the expectation values with respect to the following operators are gauge invariant: $\hat{r}$, $\hat{p}$, $\hat{L}_z = \hat{x}\hat{p}_y - \hat{y}\hat{p}_x$ and $\hat{L}_z = \hat{x}\hat{\pi}_y - \hat{y}\hat{\pi}_x$?

(b) Calculate the commutator $[\hat{\pi}_x, \hat{\pi}_y]$ and show that its value is gauge invariant, i.e. its $A$-dependency can be completely expressed in terms of $B_z$.

(c) Now we explicitly assume that $B_z = B = \text{const}$., and we introduce $\hat{\pi}_\pm = \hat{\pi}_x \pm i\hat{\pi}_y$ and the operators $\hat{a} = \frac{\hat{\pi}_x}{\sqrt{2q\hbar B}}$, $\hat{a}^\dagger = \frac{\hat{\pi}_y}{\sqrt{2q\hbar B}}$. Calculate the commutator $[a, \hat{a}^\dagger]$, express $H$ in terms of $\hat{a}$ and $\hat{a}^\dagger$ and thus show that one can map the problem onto that of a harmonic oscillator with creation and annihilation operators $\hat{a}^\dagger$, $\hat{a}$. What are the possible energy eigenvalues of the problem? What changes under the gauge-transformation $A' = A + \nabla \chi$?
Exercise 2 – Brillouin function

The energy of a spin-$J$ in a magnetic field $B$ is given by $E_{J_z} = -g\mu_B J_z B$ with the Bohr magneton $\mu_B = \frac{e\hbar}{2m}$ and the gyromagnetic factor $g$.

(a) Calculate the partition function

$$Z = e^{-\beta F} = \sum_{J_z=-J}^{+J} e^{-\beta E_{J_z}}.$$

(b) Derive the magnetization $M = -N/V \frac{\partial F}{\partial B}$ and express it in terms of the Brillouin function

$$B_J(x) = \frac{2J+1}{2J} \coth \left( \frac{2J+1}{2J} x \right) - \frac{1}{2J} \coth \left( \frac{x}{2J} \right).$$

(c) Taylor expand $B_J(x)$ for small $x$ up including terms of order $x^3$.

Exercise 3 – Critical behavior and spontaneous magnetization

To describe the phenomenon of a spontaneous magnetization, we introduce the self-consistent field $B \rightarrow B + \lambda M$ in the result from the previous exercise. $\lambda$ is a material-dependent parameter and incorporates the exchange interaction between different spins.

(a) Using the Taylor expansion from the previous exercise, find the critical Temperature $T_c$ for the appearance of a ferromagnetic phase and, by taking into account terms of order $M^3$, find the spontaneous magnetization $M(T)$ as a function of the temperature around the critical point, i.e. expand in $\delta T = T_c - T \ll T_c$.

(b) Calculate the susceptibility $\chi = \mu_0 \frac{\partial M}{\partial B}$ in the vicinity of the critical point, i.e. assume that $M$ is small and expand in $\delta T$. 