Theoretische Festkörperphysik
Sommersemester 2012
Übungsblatt 11

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Exercise 1 – Stoner model and ferromagnetic instability

We want to describe ferromagnetic order in the N-site Hubbard model with the Hamiltonian

$$H = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + U \sum_j \hat{n}_{j\uparrow} \hat{n}_{j\downarrow} - \frac{g\mu_B H}{2} \sum_j (\hat{n}_{j\uparrow} - \hat{n}_{j\downarrow}).$$

(a) The Stoner model is given by the Hamiltonian

$$\mathcal{H} = \sum_{k\sigma} (\epsilon_k + Us_{-\sigma}) \hat{n}_{k\sigma} - Us_{\uparrow} s_{\downarrow} - \frac{g\mu_B H}{2} \sum_j (\hat{n}_{j\uparrow} - \hat{n}_{j\downarrow}),$$

where $s_{\sigma} = \langle \Psi_0 | \sum_k \hat{n}_{k\sigma} | \Psi_0 \rangle$ has to be determined self-consistently and $\Psi_0$ is the ground state wave function ($T = 0$). Note that we assume $s_{\sigma}$ to be homogenous in the ground state. Derive the Stoner Hamiltonian by using the mean field approximation

$$\hat{A} \hat{B} = \hat{A} \langle \hat{B} \rangle + \langle \hat{A} \rangle \langle \hat{B} \rangle + (\hat{A} - \langle \hat{A} \rangle) (\hat{B} - \langle \hat{B} \rangle) \approx \hat{A} \langle \hat{B} \rangle + \langle \hat{A} \rangle \hat{B} - \langle \hat{A} \rangle \langle \hat{B} \rangle.$$

(b) We introduce the number $n$ of electrons and the magnetization $m$ per site of the system as $s_{\sigma} = \frac{1}{2}(n + m)$ and the density of states $N(\epsilon) = (1/N) \sum_k \delta(\epsilon - \epsilon_k)$, so that we can write $s_{\sigma} = \int^{\mu_0 + \delta\mu_\sigma(m)} d\epsilon N(\epsilon)$. We defined the chemical potential $\mu_0$ for vanishing magnetization $m = 0$ and its (small) correction $\delta\mu_\sigma(m) = O(m)$ for finite $m$. Without being interested in the specific value of $\mu_0$, what is the correction to the chemical potential $\delta\mu_\sigma$ in powers of $m$ up to second order terms $m^2$. To this end, make a Taylor expansion of the self-consistency equation around $\mu_0$ and solve for $\delta\mu_\sigma$.

(c) Calculate the energy change $\Delta E(m) = E_{\text{tot}}(m) - E_{\text{tot}}(0)$ where the total energy is $E_{\text{tot}}(m) = \langle \Psi_0 | H | \Psi_0 \rangle$ up to terms including $m^2$. Discuss the result for $\Delta E(m)$, in particular, when do you expect the appearance of ferromagnetic order in the system?

(d) Find the magnetization $m$ by minimizing the energy $\Delta E(m)$, give the result for the Stoner susceptibility $\chi = \frac{N}{V\mu_B} \frac{\partial m}{\partial H}$ and discuss the result.
Exercise 2 – London equations

In order to describe the Meissner Ochsenfeld effect (perfect diamagnetism) in a superconducting system, we use the approach put forward by F. and G. London and start with the equation

\[
\mathbf{j} = -\frac{1}{\mu_0} \frac{1}{\lambda^2} \mathbf{A}.
\]  

(1)

The intuition behind this equation is that the behavior of such a system is due to a superconducting condensate described by a macroscopic wave-function \( |\Psi_s\rangle \). In presence of an electromagnetic field, the current is then due to \( \hat{\mathbf{j}} = q\hat{\mathbf{v}} \) where the kinetic momentum is \( m\hat{\mathbf{v}} = \hat{\mathbf{p}} - q\mathbf{A} \). Taking the quantum average with respect to the ground state then gives \( \langle \Psi_s | \hat{\mathbf{j}} | \Psi_s \rangle = -i\hbar \langle \Psi_s | \nabla | \Psi_s \rangle - \frac{n_s q^2}{m} \mathbf{A} \), which equals to (1) up to a phase term \( -i\hbar \langle \Psi_s | \nabla | \Psi_s \rangle \) that plays no role in the following (the argument is that the ground state is rigid under a perturbation with \( \mathbf{A} \), contrary to the normal state, where this term almost exactly cancels the second). \( n_s \) is the density of this condensate (Cooper pairs) and we can identify the parameter as \( \mu_0 \lambda^2 = \frac{m}{n_s q^2} \).

(a) Derive the first and second London equations:

\[
\frac{\partial \mathbf{j}}{\partial t} = \frac{1}{\mu_0} \frac{1}{\lambda^2} \mathbf{E},
\]

\[
\nabla \times \mathbf{j} = -\frac{1}{\mu_0} \frac{1}{\lambda^2} \mathbf{B}.
\]

(b) Use the London equations together with the Maxwell equations to derive a differential equation for the magnetic field \( \mathbf{B}(x) \) inside the superconducting region.

(c) Solve this differential equation with the proper boundary conditions at the interface and deep inside the superconductor. Plot and discuss your solution. What currents are flowing? Why do they not decay? Can you give the parameter \( \lambda \) a direct physical meaning and estimate it’s size? How does his finding explain the perfect diamagnetism of a macroscopic superconductor.