Exercise 1 – Thomson-effect

Within the framework of Drude, we want to calculate the power dissipated by electrons in a metal with an applied electric field $E$ in addition to a thermal gradient $\nabla T$.

(a) First, we consider an electron subject to a collision at location $r$, and which was previously scattered at location $r - d$. Analogously to the lecture, assume that the time between these two scattering events is given by the scattering time $\tau$. Specify $d$ in the presence of the electric field $E$.

(b) Disregarding the thermal gradient for now, what is the energy gained by the electron in the electric field? What is the power dissipated due to this scattering event? Average over the velocity distribution of all electrons and show that the total power dissipated is given by $P_E = \sigma E^2$, where $\sigma$ is the Drude conductivity.

(c) Now, we want to find out the additional dissipation due to the thermal gradient. To this end, assume that the energy of an electron is given by $\epsilon(T(r))$ and that the temperature varies only weakly on the scale given by $d$, so that you can perform a Taylor expansion. Then perform the average over all electron velocities and show that the dissipated power to lowest order in $E$ and $\nabla T$ reads

$$P_{E,T} = \frac{n e \tau}{m} \frac{d \epsilon}{d T} (E \cdot \nabla T) .$$

Exercise 2 – Helicon waves

We consider a metal in a homogenous magnetic field along the $z$-Axis, $B = B \hat{z}$.

(a) We want to study circularly polarized excitations in this system of the form $E e^{-i\omega t}$, $E_y = iE_x$ und $E_z = 0$. Show that within the framework of Drude, the relation between the current and the AC electric field is:

$$j_x = \frac{\sigma_0}{1 - i\tau(\omega - \omega_c)} E_x , \quad j_y = ij_x , \quad j_z = 0 .$$
Remark: The magnetic component of these excitations is as usual much smaller than the electric component and therefore can be neglected.

(b) Use the Maxwell equations together with the expression for the current from (a) to show that there exists a solutions of the form

\[ E_x = E_0 e^{i(kz - \omega t)}, \quad E_y = +iE_x, \quad E_z = 0, \]

provided that \( k^2c^2 = \epsilon(\omega) \omega^2 \), where

\[ \epsilon(\omega) = 1 - \frac{\omega_p^2}{\omega} \left( \frac{1}{\omega - \omega_c + i/\tau} \right). \]

(c) In experiments, we typically have \( \omega_p \gg \omega_c \) and \( \omega_c \tau \gg 1 \). Discuss \( \epsilon(\omega) \) and show that there exist solutions for arbitrary \( k \) for frequencies \( \omega < \omega_c \) and \( \omega > \omega_p \).

(d) Determine the dispersion relation \( \omega(k) \) for the low-frequency modes (helicon waves), i.e. \( \omega \ll \omega_c \).

Exercise 3 – Free electron gas in two dimensions

We consider a system in two dimensions consisting of non-interacting, free electrons with the dispersion relation \( \epsilon_k = \frac{\hbar^2 k^2}{2m} \).

(a) Give the relation between the electron density \( n \) and the Fermi-wavevector \( k_F \) (at \( T = 0 \)).

(b) Calculate the density of states \( N(\epsilon) \) and show, that \( N(\epsilon < 0) = 0 \) and \( N(\epsilon > 0) = \text{const.} \)

(c) We now want to calculate \( n(\mu) \) for finite temperatures \( T > 0 \) using the result from part (b). First, consider the Sommerfeld expansion, and show that most of the terms of this expansion vanish. Summing all the terms in the Sommerfeld expansion, show that \( \mu = \epsilon_F \) for arbitrary temperature.

(d) Perform the exact integration of the formula for the density, \( n(\mu) = \int_{-\infty}^{+\infty} N(\epsilon) f(\epsilon) d\epsilon \), and show that

\[ \epsilon_F = \mu + k_B T \ln \left( 1 + e^{-\mu/k_B T} \right). \]

What is the mathematical origin for the discrepancy to the result of (c)? Estimate the order of magnitude of this correction and discuss its relevance.

Remark:

\[ \int \frac{dx}{1 + e^x} = x - \ln(1 + e^x) \]